Inequalities

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Let a,b,c be non-negative real numbers such that

$a^2+b^2+c^2+abc=4$, prove that

$0 \le ab+bc+ca-abc \le 2.$

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First note that $0 \le ab+bc+ca-abc \le 2 \Leftrightarrow 0 \le ab+bc+ca+a^2+b^2+c^2-4 \le 2 \Leftrightarrow$

(**1**) $4 \le ab+bc+ca+a^2+b^2+c^2 \le 6$

Also note that $a^2+b^2+c^2+abc=4$ and $a,b,c \ge 0$ implies $a,b,c \le 2.$ But since the inequality obviously holds if at least one of numbers a,b,c equal 2 then for further we assume that $a,b,c<2.$ Furthermore, if at least one of numbers a,b,c equal $0,$ let it be c then $a^2+b^2=4$ and $ab+bc+ca-abc=ab,$ where $ab \ge 0$ and $ab \le \dfrac{a^2+b^2}{2}=\dfrac{4}{2}=2.$

Thus, we can assume that $a,b,c \in (0,2).$

Using substitution $a=2\cos\alpha, b=2\cos\beta, c=2\cos\gamma,$ where $\alpha,\beta,\gamma \in (0,\pi/2),$ we can equivalently rewrite the constraint and the inequality, respectively, as follows:

(**2**) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1$

(**3**) $1 \le \cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos^2\alpha+\cos^2\beta+\cos^2\gamma \le \dfrac{3}{2}$

Since for $\alpha,\beta,\gamma \in ((0,\pi/2)$ equation (**1**) is equivalent to $\alpha+\beta+\gamma=\pi$ we can consider α,β,γ as angles of some non-obtuse tiangle with correspondent sidelenghts $a,b,c.$

Let s,R,r be semiperimeter, circumradius and inradius of this triangle.

Then $\cos\alpha+\cos\beta+\cos\gamma=1+\dfrac{r}{R},\ \cos\alpha\cos\beta\cos\gamma=\dfrac{s^2-(2R+r)^2}{4R^2}.$

Hence, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1-\dfrac{s^2-(2R+r)^2}{2R^2}, \cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha=$ $\dfrac{1}{2}\left(\left(1+\dfrac{r}{R}\right)^2-\left(1-\dfrac{s^2-(2R+r)^2}{2R^2}\right)\right)=\dfrac{r^2+s^2}{4R^2}-1$ and, therefore,

$\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\dfrac{r^2+s^2}{4R^2}-\dfrac{s^2-(2R+r)^2}{2R^2}=$ $\dfrac{8R^2+8Rr+3r^2-s^2}{4R^2}$. Thus, inequality (**2**) becomes $1 \le \dfrac{8R^2+8Rr+3r^2-s^2}{4R^2} \le \dfrac{3}{2}.$

Using Gerretsen's inequalities $s^2 \le 4R^2+4Rr+3r^2$ we obtain

$$\frac{8R^2+8Rr+3r^2-s^2}{4R^2} \ge \frac{8R^2+8Rr+3r^2-(4R^2+4Rr+3r^2)}{4R^2}=\frac{1}{R}(R+r)>1.$$

Using Walker's inequality for an acute triangle $2R^2+8Rr+3r^2 \le s^2$ we obtain

$$\frac{8R^2+8Rr+3r^2-s^2}{4R^2} \le \frac{8R^2+8Rr+3r^2-(2R^2+8Rr+3r^2)}{4R^2}=\frac{6R^2}{4R^2}=\frac{3}{2}.$$